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Robust Stability of Systems with Delayed Feedback*

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Abstract. Some issues in the stability of differential delay systems in the linear and the nonlinear case are investigated. In particular, sufficient robustness conditions are derived under which a system remains stable, independent of the length of the delay(s). Applications in the design of delayed feedback systems are given. Two approaches are presented, one based on Lyapunov theory, the other on a transformation to Jordan form. In the former, sufficient conditions are obtained in the form of certain Riccati-type equations.

1. Introduction

In many applications, such as man-machine systems, biomedical systems, process control, remote control and robotics, delays are inherent in the control due to transportation lags, and conduction or communication times. Moreover, the delay may not be exactly known, or even fixed. The purpose of this paper is to investigate some stabilization issues of such delay systems, in particular their robustness with respect to the delay times. A connection is made to the theory of singular systems, which may provide some new insights into the regularization of such systems.

In order to fix the ideas, we assume that some system needs to be regulated about some fixed operating point. Locally, then, the system dynamical equations are suitably approximated by a linear system,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t)$ in (1) denotes the state excursion, away from the nominal operating point, and $u(t)$ is an admissible control. In order to accommodate for transportation and/or communication lags, we assume that at time t the admissible

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controls belong to the space generated by $\{x(\sigma) \mid \sigma \leq t - r\}$ for some $r > 0$. In particular, the case where $u(t)$ is some nonlinear function of the state at the latest available instant is considered here, as it coincides with the design of a state feedback controller in the absence of a delay:

$$u(t) = K[x(t - r)]. \tag{2}$$

Such more general nonlinear control functions are for instance desired in the case of bounded admissible inputs, leading to linearly or quadratically saturating controls [1].

The closed loop system is then given by the delay differential system

$$\dot{x}(t) = Ax(t) + F[x(t - r)] \tag{3}$$

where $F(\cdot) = BK(\cdot)$. It is well known that the system can also be represented via time scaling by

$$\epsilon\dot{x}(t) = Ax(t) + F[x(t - 1)] \tag{4}$$

where $\epsilon = 1/r$. One of the approaches used is to consider the above equation as a singular perturbation of the difference equation with continuous time,

$$Ax(t) = -F[(x(t - 1))], \tag{5}$$

and from this the map $-A^{-1}F: R^n \to R^n$, with its corresponding dynamical system [2]. In the scalar case with $F(x) = bx$, the following results are known for the differential delay system [3], [4]: If $a < 0$, then all roots of the characteristic equation $s = a + be^{-rs}$ have real part smaller than some number for all $r > 0$. If $a > 0$, then for every real $\sigma > 0$, there exists an $r_0(\sigma)$ such that the characteristic equation has at least one root with real part larger than σ for all $r > r_0(\sigma)$.

This paper is organized as follows: in Section 2, we consider the class of linear differential delay systems. Sufficient conditions for robust stability, in terms of Riccati-like equations, are established based on the Lyapunov theory. In Section 3 a direct approach is used for the nonlinear and linear robust stability problem.

2. Riccati-type equations

We consider here the equation (3) for the case of a linear feedback (2), but with arbitrary delay. Let for simplicity the equation be rewritten as

$$\dot{x}(t) = Ax(t) + Bx(t - r). \tag{6}$$

The following sufficient condition is readily established.

Theorem 1. *The system* (6) *is asymptotically stable, if there exists a triple of positive definite* (*symmetric*) *matrices* P, Q *and* R *such that*

$$A'P + PA + Q + PBQ^{-1}B'P + R = 0. \tag{7}$$

Proof. Consider the function

$$V(x) = x'Px + \int_{t-r}^{t} x'(\sigma)Qx(\sigma)d\sigma. \tag{8}$$

Along trajectories of (6) we have

$$\begin{aligned}\dot{V}(t) = &-\big[x(t-r)'Q^{1/2} - x(t)'PBQ^{-T/2}\big]\big[Q^{T/2}x(t-r) - Q^{-1/2}B'Px(t)\big]\\ &+ x(t)'\big[A'P + PA + Q + PBQ^{-1}B'P\big]x(t)\\ &\leq -x(t)'Rx(t) \leq 0. \end{aligned}\tag{9}$$

By Lyapunov's lemma, global asymptotic stability follows if the conditions of the theorem are satisfied. □

The left hand side of the equation (7) is similar to the Riccati equation, but has a sign change in the quadratic term. Some other sufficient conditions can be derived from it.

Corollary 1.1. *The system* (6) *is asymptotically stable, if there exists a positive definite* (*symmetric*) *matrix* Z *and a positive scalar* α *such that*

$$Z + \alpha(ZA' + AZ) + \alpha^2 BZB' < 0. \tag{10}$$

Proof. Set $P = pZ^{-1}$ and $Q = qZ^{-1}$ with $\frac{p}{q} = \alpha$ in Theorem 1. □

It is an easy consequence of (10) that in the special case of $B = bI_n$ with b scalar, the robust stability is guaranteed if $\operatorname{Re}\lambda(A) < -|b|$. This occurs in the linearized dynamics of a Continuous Stirred Tank Reaction, CSTR [5]. A particularly useful sufficient condition is given in the following.

Corollary 1.2. *If the symmetric part* A_s *of* A *satisfies*

$$A_s \leq -\frac{1}{2}(qI + BB'/q) \tag{11}$$

for some positive scalar q, *then the system* (6) *is asymptotically stable.*

Proof. Set $P = I$ and $Q = qI$ in Theorem 1. □

Note that the choice $q = \|B\|$ leads to a simple sufficient condition

$$A_s \leq -\frac{1}{2}\big(\|B\|I + BB'/\|B\|\big)$$

which reduces in the scalar case to the well known criterion $a \leq -\|b\|$. This sufficient condition is further implied by $A_s \leq -\|B\|I$, since $\frac{1}{\|B\|}BB' \leq \|B\|I$.

This Lyapunov method is easily generalizable to systems with multiple delays. We state the following result.

Theorem 2. *The system*

$$\dot{x}(t) = Ax(t) + B_1x(t - \tau_1) + B_1x(t - \tau_2) + \cdots + B_mx(t - \tau_m). \tag{12}$$

is asymptotically stable for all values of the delays $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ *if there exists a (symmetric) positive definite matrix P, and (symmetric) positive definite matrices* $Q_1 \geq Q_2 \geq \cdots \geq Q_m \geq Q_{m+1} = 0$ *such that*

$$A'P + PA + Q_1 + \sum_{i=1}^{m} PB(Q_i - Q_{i+1})^{-1}B'P < 0. \tag{13}$$

Proof. Consider the function

$$V(x) = x'Px + \int_{t-\tau_1}^{t} x'(\sigma)Q_1x(\sigma)d\sigma + \int_{t-\tau_2}^{t-\tau_1} x'(\sigma)Q_2x(\sigma)d\sigma + \cdots + \int_{t-\tau_m}^{t-\tau_{m-1}} x'(\sigma)Q_mx(\sigma)d\sigma \tag{14}$$

and use the same trick of "completing the squares" as in the proof of Theorem 1. □

Without any problem, Corollaries 1 and 2 are easily generalized for this case as well.

3. Approach via Jordan form

Consider a system of linear differential delay equations

$$\epsilon\dot{x}(t) = Ax(t) + Bx(t - 1) \tag{15}$$

where $x \in \mathbf{R}^n$, A and B are real $n \times n$ matrices, and ϵ is a positive parameter. By a similarity, the system (15) is equivalent to one with an A-matrix in Jordan canonical form, $A = \text{Blockdiag}(J_i)$.

Theorem 3. *If all eigenvalues of A have negative real parts, and all eigenvalues of* $A^{-1}B$ *are inside the unit circle, then the null solution of the system* (15) *is stable for all sufficiently small* $\epsilon > 0$.

Proof. By setting $\epsilon = 0$ in (15), the limiting difference system $-Ax(t) = Bx(t-1)$ is obtained, which is equivalent to (A is nonsingular)

$$x(t) = -A^{-1}Bx(t - 1). \tag{16}$$

The conditions of the theorem imply that the null solution of (16) is asymptotically stable. We shall show that the solutions of system (16) and system (15) are close within finite time intervals for small $\epsilon > 0$. More precisely, let B_1 be a unit ball in $C([-1, 0], \mathbf{R})$, and let $x_\varphi(t)$ denote the solution of (16) through the initial function φ, while $x^\epsilon_\psi(t)$ is the solution of (15) through ψ with fixed $\epsilon > 0$. Then we shall show that for arbitrary fixed $T > 0$ and $\sigma > 0$ there exists a $\delta > 0$ and an $\epsilon_0 > 0$ such that for every $\varphi, \psi \in C([-1,0], \mathbf{R})$ one has

$$\sup\{|x_\varphi(t) - x^\epsilon_\psi(t)|,\ t \in [0,T]\} \leq \sigma \tag{17}$$

for all $0 < \epsilon \leq \epsilon_0$, provided $\sup\{|\varphi(t) - \psi(t)|,\ t \in [-1,0]\} \leq \delta$. From (17) and the asymptotic stability of the null solution of (16), the asymptotic stability of the null solution of (15) is straightforward (for $0 < \epsilon \leq \epsilon_0$). So we are left to prove the closeness mentioned above.

Consider first the case of a real eigenvalue $\lambda_1 < 0$. Let it correspond to a Jordan block J_1 of size m_1 of matrix A. Equivalently, consider the m_1 equations of systems (15):

$$\begin{aligned}
\epsilon\dot{x}_1(t) &= \lambda_1 x_1(t) + x_2(t) + b_{11}x_1(t-1) + \cdots + b_{1n}x_n(t-1) \\
\epsilon\dot{x}_2(t) &= \lambda_1 x_2(t) + x_3(t) + b_{21}x_1(t-1) + \cdots + b_{2n}x_n(t-1) \\
&\ \vdots \\
\epsilon\dot{x}_{m_1-1}(t) &= \lambda_1 x_{m_1-1}(t) + x_{m_1}(t) + b_{m_1-1}x_1(t-1) + \cdots + b_{m_1-1,n}x_n(t-1) \\
\epsilon\dot{x}_{m_1}(t) &= \lambda_1 x_{m_1}(t) + b_{m_1,11}x_1(t-1) + \cdots + b_{1m}x_m(t-1)
\end{aligned} \tag{18}$$

where $\lambda_1 < 0$.

From the last equation of system (18), closeness between the m_1-th components of systems (16) and (15) is derived in essentially the same way as it is done in [2]. Substituting then $x_{m_1}(t)$ to the $(m_1 - 1)$-st equation of (18), we derive in the same way the closeness between the $(m_1 - 1)$-st components of systems (16) and (15), and so on up to the first equation of (18). The arguments for other blocks J_i, for $i > 1$ associated with real eigenvalues of the matrix A are the same.

To show the closeness in the case of complex conjugate eigenvalues of matrix A we consider first the simplest situation of $m = 1$. Therefore, we may assume that (notice the rescaling)

$$A = \begin{bmatrix} -1 & -\omega \\ \omega & -1 \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}. \tag{19}$$

The limit case $\epsilon = 0$ corresponds to the system (16) with the consistency condition

$$x(0) = -A^{-1}Bx(-1), \tag{20}$$

where $x = [x_1, x_2]'$. We show the closeness componentwise. The first component of (15) is given by

$$\begin{aligned}
x^\epsilon_1(t) &= e^{-t/\epsilon}\left(x^0_1 \cos\frac{\omega t}{\epsilon} - x^0_2 \sin\frac{\omega t}{\epsilon}\right) \\
&+ \frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}\left[b_{11}\cos\frac{\omega(s-t)}{\epsilon} + b_{21}\sin\frac{\omega(s-t)}{\epsilon}\right] x_1(s-1)ds
\end{aligned}$$

$$+\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}\left[b_{12}\cos\frac{\omega(s-t)}{\epsilon}+b_{22}\sin\frac{\omega(s-t)}{\epsilon}\right]x_2(s-1)ds, \quad (21)$$

and the first component of (16) is given by

$$x_1(t)=\frac{1}{1+\omega^2}\left[(b_{11}-\omega b_{21})x_1(t-1)+(b_{12}-\omega b_{22})x_2(t-1)\right], \quad (22)$$

where $t\in[0,1]$ and $x_1(t-1)$, $x_2(t-1)$ are the first and second components of the initial function in $[-1,0]$, $x_1^0=x_1(0)$, $x_2^0=x_2(0)$. The first component of the consistency condition (20) takes the form

$$x_1(0)=\frac{1}{1+\omega^2}\left[(b_{11}-\omega b_{21})x_1(-1)+(b_{12}-\omega b_{22})x_2(-1)\right]. \quad (23)$$

Assuming differentiability of $\phi(s)$, $s\in[-1,0]$, it is an easy exercise to find that

$$\begin{aligned}&\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}\cos\frac{\omega(s-t)}{\epsilon}\phi(s)ds\\ &\quad=\frac{\phi(t)}{1+\omega^2}-\frac{\phi(0)}{1+\omega^2}e^{-t/\epsilon}\left[\cos\frac{\omega t}{\epsilon}-\omega\sin\frac{\omega t}{\epsilon}\right] \quad (24)\\ &\quad-\frac{1}{1+\omega^2}\int_0^t e^{\frac{(s-t)}{\epsilon}}\left[\cos\frac{\omega(s-t)}{\epsilon}+\omega\sin\frac{\omega(s-t)}{\epsilon}\right]\phi'(s)ds,\end{aligned}$$

and likewise

$$\begin{aligned}&\frac{1}{\epsilon}\int_0^t e^{\frac{s-t}{\epsilon}}\sin\frac{\omega(s-t)}{\epsilon}\phi(s)ds\\ &\quad=-\frac{\omega\phi(t)}{1+\omega^2}+\frac{\phi(0)}{1+\omega^2}e^{-t/\epsilon}\left[\omega\cos\frac{\omega t}{\epsilon}+\sin\frac{\omega t}{\epsilon}\right] \quad (25)\\ &\quad-\frac{1}{1+\omega^2}\int_0^t e^{\frac{(s-t)}{\epsilon}}\left[\sin\frac{\omega(s-t)}{\epsilon}-\omega\cos\frac{\omega(s-t)}{\epsilon}\right]\phi'(s)ds.\end{aligned}$$

We note next that

$$\begin{aligned}&\int_0^t e^{\frac{s-t}{\epsilon}}\sin\frac{\omega(s-t)}{\epsilon}\phi'(s)ds=O(\epsilon),\\ &\int_0^t e^{\frac{s-t}{\epsilon}}\cos\frac{\omega(s-t)}{\epsilon}\phi'(s)ds=O(\epsilon), \quad (26)\end{aligned}$$

as $\epsilon\longrightarrow+0$. Therefore: substituting (25) and (26) into $|x_1^\epsilon(t)-x_1(t)|:=\Delta(t)$, $t\in[0,1]$, and taking (23) into account, we obtain $\Delta(t)=O(\epsilon)$ for $t\in[0,1]$, and $\epsilon\ll 1$. The same arguments show the closeness for the respective second components. To show the closeness in the case of continuous initial functions $x(s)$, $s\in[-1,0]$, the approach of [2, Chapter 3] may be used with more technical details involved.

The arguments for $2m$-dimensional A blocks corresponding to a Jordan block of size m for a complex eigenvalue (and its conjugate) is treated similarly but with more technical computations. □

A partial converse is given in the following.

Theorem 4. *If there exist eigenvalues of A with positive real part, then the null solution is unstable for sufficiently small $\epsilon > 0$.*

Proof. The proof will be restricted to the case of one simple positive (real) eigenvalue, with all other having negative real parts, and the case of one simple complex conjugate pair with positive real part, and all other eigenvalues having a negative real part. The proof in the other cases proceeds with the same ideas, but has more complicated details.

In the Jordan canonical form we may assume thus that $A_1 = \lambda_1 > 0$, and $\text{Re}\,\lambda_i < 0$ for $i > 1$. Next we shall make use of the following known fact. The scalar transcendental equation

$$\epsilon s = a + b\exp(-s) \tag{27}$$

has roots with i) uniformly (with respect to $\epsilon > 0$) bounded from above real parts in the case $a < 0$, and ii) with unbounded from above real parts as $\epsilon \longrightarrow +0$ in the case $a > 0$ [3]. Also system (15) is equivalent to the system

$$\dot{z}(t) = Az(t) + Bz(t-r) \tag{28}$$

with $r = \frac{1}{\epsilon}$, and (27) is equivalent to

$$s = a + b\exp(-rs). \tag{29}$$

Consider now the corresponding characteristic equation

$$\Delta(s) = \det\left[sI - A - B\exp(-sr)\right] = 0 \tag{30}$$

or in expanded form:

$$\det\begin{bmatrix} s-\lambda_1 - b_{11}e^{-sr} & -b_{12}e^{-sr} & \cdots & \cdots & -b_{1n}e^{-sr} \\ -b_{21}e^{-sr} & s-\lambda_2 - b_{22}e^{-sr} & -1-b_{23}e^{-sr} & \cdots & -b_{2n}e^{-sr} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{n1}e^{-sr} & \cdots & \cdots & -b_{n,n-1}e^{-sr} & s-\lambda_m - b_{nn}e^{-sr} \end{bmatrix} = 0\,\cdot$$

Calculating the above determinant by the Laplace expansion using the first row, we have

$$\Delta(s) = \left(s - \lambda_1 - b_{11}e^{-sr}\right)P(s) + Q(s) = 0 \tag{31}$$

where $|Q(s)| \longrightarrow 0$ as $r \longrightarrow +\infty$ and $\text{Re}\, s \geq 1$, and $P(s)$ has no zeros and is bounded away from zero for $\text{Re}\, s \geq r_0 > 0$ for some r_0 since all λ_i, $i \geq 2$ have negative real parts. So (31) may be rewritten in the form

$$\Delta(s) = s - \lambda_1 - b_{11}e^{-sr} + R(s) = 0,$$

where $|R(s)|$ is sufficiently small as $r \gg 1$ and $\text{Re}\, s > 1$.

Finally, Rouché's Theorem is invoked, guaranteeing the existence of close zeros of analytic functions under small perturbations. Indeed, since $\lambda_1 > 0$, by the above equation, $\Delta_1(s) = s - \lambda_1 - b_{11}e^{-sr}$ has a zero s_0 with unbounded from above real part as $r \longrightarrow +\infty$. The perturbation $R(s)$ is small in the vicinity of this zero since $\text{Re}\, s_0 \gg 1$ and $r \gg 1$. Therefore, in the neighborhood of s_0 there exists a zero of $\Delta(s)$.

In the case of a complex conjugate pair with positive real part, we again focus attention to the subsystem, with $\epsilon > 0$,

$$\epsilon\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & \omega \\ -\omega & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}. \tag{32}$$

We claim that the real parts of the zeros of the corresponding characteristic equation of (32) are unbounded from above for $\epsilon > 0$. Indeed, the characteristic equation has the form

$$\det\begin{bmatrix} \epsilon s - 1 - b_{11}e^{-s} & -\omega - b_{12}e^{-s} \\ \omega - b_{21}e^{-s} & \epsilon s - 1 - b_{22}e^{-s} \end{bmatrix} = 0, \tag{33}$$

or

$$\begin{aligned} 0 = (\epsilon s - 1)^2 + \left[-(\epsilon s - 1)(b_{11} + b_{22}) + \omega(b_{21} - b_{12})\right]e^{-s} \\ + (b_{11}b_{22} - b_{12}b_{21})e^{-2s}. \end{aligned} \tag{34}$$

Now we note that $P(s)$ defined as $\epsilon s + a + be^{-s}$ has zeros with real parts uniformly bounded from above for $\epsilon > 0$ in the case $a > 0$ (with the opposite statement in the case $a < 0$). Therefore, the same property is enjoyed by the quasipolynomial $Q(s) = P^2(s) + c$, where c is a constant. But $Q(s)$ is the right hand expression in (34) for an appropriate c. □

Extensions exist to the nonlinear case with $A = -I_n$. In particular the established results relate to the invariance and the continuity properties: The following theorem says that the set $C([-1,0], D)$ is invariant under the semiflow defined by (4), provided D is convex, closed, and invariant under F.

Theorem 5. *If D is a closed convex invariant domain under the map F, then for any $\varphi \in C([-1,0], D) \stackrel{\text{def}}{=} X_D$ the solution to the singular delay equation has the property that $x_\varphi^\epsilon \in D$ for all $t \geq 0$ and $\epsilon \geq 0$.*

Proof. Let $D \subseteq X = \mathbf{R}^n$ be a convex and closed domain which is invariant under F. Take $\phi \in X_D = \{\phi \in X \mid \phi(t) \in D, \forall t \in [-1,0]\}$ and suppose that there exists a time $t_0 \geq 0$ such that $x_\phi^\epsilon(t) \in \partial D$ (the boundary of D) and the solution $x_\phi^\epsilon(t)$ leaves the domain D for $t \geq t_0$. Then the vector $\dot{x}_\phi^\epsilon(t_0)$ is directed outside the domain D, and so is the vector $\epsilon\dot{x}_\phi^\epsilon(t_0)$. Since we may assume t_0 to be the first point at which $x_\phi^\epsilon(t)$ leaves the domain D, the $F[x_\phi^\epsilon(t_0 - 1)]$ lies inside the domain D. Therefore the vector $F[x_\phi^\epsilon(t_0 - 1)] - x_\phi^\epsilon(t_0)$ is directed inside the domain D. But according to Equation (4) for $A = -I$:

$$\epsilon\dot{x}_\phi^\epsilon(t_0) = F[x_\phi^\epsilon(t_0 - 1)] - x_\phi^\epsilon(t_0) \ :$$

a contradiction. This proves the theorem. □

Consider the nonlinear equation

$$\epsilon\dot{x}(t) = -x(t) + F[x(t-1)]. \tag{35}$$

Suppose that the multidimensional map $F: \mathbf{R}^n \to \mathbf{R}^n$ has an arbitrary fixed point x_0 such that there exists an open convex neighborhood $\mathcal{U}$ of x_0 where $F^k(\mathcal{U})$ are convex sets. (Therefore $\bigcap_{k\geq 0} F^k(\mathcal{U}) = x_0$, the fixed point is locally attracting.) Let $X_{\mathcal{U}} = C([-1,0],\mathcal{U})$ be a subset of initial functions and x^ϵ_ϕ be a solution of (35) constructed through $\phi \in X_{\mathcal{U}}$.

Theorem 6. *For arbitrary positive ϵ one has*

$$\lim_{t\to\infty} x^\epsilon_\phi(t) = x_0, \quad \forall \phi \in X_{\mathcal{U}}.$$

The theorem says that the (locally) attracting fixed point is asymptotically stable. To facilitate the proof of Theorem 6, we first state a technical lemma.

Lemma. *Take an arbitrary open convex set $\mathcal{U}$ containing a domain D_1 and contained in domain D, and arbitrary initial data $\phi \in X_D$. If $\phi(0)$ is in* cl$\mathcal{U}$ *the closure of $\mathcal{U}$, then $x^\epsilon_\phi(t)$ is in the closure of $\mathcal{U}$ for all $t \geq 0$. If $\phi(0)$ is not in the closure of $\mathcal{U}$ then there exists a time $t_0 = t_0(\phi, D)$ such that $x^\epsilon_\phi(t_0) \in \partial\mathcal{U}$ (the boundary of U) and $x^\epsilon_\phi(t) \in$ cl$\mathcal{U}$ for all $t \geq t_0$.*

Proof. Suppose first that $\phi(0) \in$ cl$\mathcal{U}$. Since cl$\mathcal{U}$ is a convex set, $\mathcal{U} \subset D$ and $\mathcal{U} \supset f(D)$, then $f(\text{cl}\mathcal{U}) \subseteq \text{cl}\mathcal{U}$. Therefore, the condition $x^\epsilon_\phi(t) \in \text{cl}\mathcal{U}$ $\forall t \geq 0$ is implied in this case from the invariance property Theorem 5.

Suppose next that $\phi(0) \notin$ cl$\mathcal{U}$. Note that $x^\epsilon_\phi(t) \in D$ for all $t \geq 0$ because of Theorem 5. So if we have the first time t_0 such that $x^\epsilon_\phi(t_0) \in \partial D$, then $x^\epsilon_\phi(t) \in$ cl$\mathcal{U}$ for all $t \geq t_0$. To show this we just have to repeat the argument of the proof of Theorem 5.

Thus suppose that $\phi(0) \notin$ cl$\mathcal{U}$ and $x^\epsilon_\phi(t) \notin$ cl$\mathcal{U}$ for all $t \geq 0$. Let V be the maximal open convex set containing $\mathcal{U}$ and such that $V \cap \text{cl}\{x^\epsilon_\phi(t), t \geq 0\} = \emptyset$. Note that V may coincide with U. Since V is the maximal set in the above sense there exists a sequence $\{t_n\}$, $n = 1, 2, \ldots$ such that $x^\epsilon_\phi(t_n) \to x_0$, where $x_0 \in \partial V$. Consider now the bound vectors $\vec{s}_n = \frac{1}{\epsilon}\left[-x^\epsilon_\phi(t_n) + F[x^\epsilon_\phi(t_n - 1)]\right]$. Since $V \subset D$, one has that $F(V) \subseteq D_1$. Therefore, all $\vec{s}_n$ are of some length bounded away from zero and are directed strictly inside the domain V. Since the origins of the bound vectors $\vec{s}_n$ converge to the point $x_0 \in \partial V$ and $\vec{s}_n$ is the vector tangent to the solution $x^\epsilon_\phi(t)$ at $t = t_n$, and $|\vec{s}_n| \geq \delta$ for some $\delta > 0$, there exists t_N such that $x^\epsilon_\phi(t') \in V$ for some $t' > t_N$. In other words, the solution $x^\epsilon_\phi(t)$ should enter the domain V, a contradiction. This completes the proof of the lemma. □

Proof of Theorem 6. Since $x = x_0$ is an attracting fixed point of the map F there exists a sequence of nested convex open neighborhoods $\mathcal{U}_k \subset \mathcal{U}_{k+1}$ such that $F(\mathcal{U}_k) \supseteq \mathcal{U}_{k+1}$ and $\bigcap_{k\geq 1} \mathcal{U}_k = x_0$ (one may choose $\mathcal{U}_{k+1} = \text{span}\{F(\mathcal{U}_k)$, $k = 1, 2, \ldots\}$). By repeated application of the lemma one sees that there exists a sequence $\{t_n\} \to \infty$ such that either $x^\epsilon_\phi(t_n) \in \partial\mathcal{U}_n$ or $x^\epsilon_\phi(t_n) \in \text{int}\mathcal{U}_n$. Because of the invariance property (Theorem 5) one has $x_\phi^\epsilon(t) \in \text{cl}\mathcal{U}_n$ for all $t \geq t_n$. Since $\bigcap_{n\geq 1} \mathcal{U}_n = x_0$, the stability follows. □

Further interesting problems remain. For instance, if the matrix A is singular, the singular perturbation $\epsilon = 0$ yields a singular nonlinear system, with the associated map implicitly defined by

$$Ax_{k+1} = -F[x_k]. \tag{36}$$

The behavior of the nonperturbed solution is not yet completely understood in this case. We remark however that for the linear case, it follows from Theorem 3 that the system is unstable for sufficiently small ϵ. This is essentially a non-robustness result of singular systems as "models" for certain differential delay systems.

4. Conclusions

Regular systems with delayed feedback lead to differential delay models. A connection between the differential delay system and a singular system was given. Robust (with respect to the delay) stability and instability conditions were presented. We have also shown that Riccati-like equations result from the robust stability criteria. Obviously, if an open loop system is unstable, then the closed loop with delayed feedback will be unstable for sufficiently large delay time. Hence robust stabilization of unstable open loop systems is impossible. This is intuitively clear since for $r \to \infty$ the closed loop properties are determined by the open loop behavior.

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